ON PRIMES REPRESENTED BY QUADRATIC POLYNOMIALS

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Abstract. This is a survey article on the Hardy-Littlewood conjecture about primes in quadratic progressions. We recount the history and quote some results approximating this hitherto unresolved conjecture.

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1. The Conjecture

It is attributed to Dirichlet that any linear polynomial with integer coefficients represents infinitely many primes provided the coefficients are co-prime. The next natural step seems to be establishing a similar statement for quadratic polynomials. G. H. Hardy and J. E. Littlewood [20] gave the following conjecture in 1922 based on their circle method.

Conjecture. *Suppose a, b and c are integers with $a > 0$, $\gcd(a,b,c) = 1$, $a+b$ and c are not both even, and $D = b^2 - 4ac$ is not a square. Let $P_f(x)$ be the number of primes $p \leq x$ of the form $p = f(n) = an^2 + bn + c$ with $n \in \mathbb{Z}$. Then*

$$P_f(x) \sim \gcd(2, a+b) \frac{\mathfrak{S}(D)}{\sqrt{a}} \frac{\sqrt{x}}{\log x} \prod_{\substack{p|a, p|b \\ p>2}} \frac{p}{p-1}, \tag{1.1}$$

where

$$\mathfrak{S}(D) = \prod_{\substack{p \nmid a \\ p>2}} \left(1 - \frac{\left(\frac{D}{p}\right)}{p-1} \right). \tag{1.2}$$

Here and after, $\left(\frac{D}{p}\right)$ denotes the Legendre symbol, i.e. its value is 1 if D is a quadratic residue modulo p, -1 if D is a quadratic non-residue modulo p and 0 if p divides D.

The conjecture has thus far resisted attack to the extent that its simplest case for the polynomial n^2+1 is not even resolved. Indeed, no polynomial of degree two or higher is known to represent infinitely many primes.

In a related problem, L. Euler and A.-M. Legendre were the first to observe that $n^2 + n + 41$ is prime for all $0 \leq n \leq 39$. G. Rabinowitsch [36] showed that $n^2 + n + A$ is prime for $0 \leq n \leq A - 2$ if and only if $4A - 1$ is square-free and the ring of integers of the number field $\mathbb{Q}(\sqrt{1-4A})$ has class number one. This question was further studied by A. Granville and R. A. Mollin in [19] and the works, particularly those of Mollin, referred to therein. It is most note-worthy that an upper bound for $P_f(x)$ of the order of magnitude predicted by (1.1) was proved in [19] unconditionally uniform in f, and uniform in x under the Riemann hypothesis for the Dirichlet L-function $L(s, (D/\cdot))$. Furthermore, it was shown unconditionally in [19] that for large R and N with $R^{\varepsilon} < N < \sqrt{R}$,

$$\#\left\{ n \leq N : n^2 + n + A \in \mathbb{P} \right\} \asymp L\left(1, \left(\frac{1-4A}{\cdot}\right)\right)^{-1} \frac{N}{\log N}$$

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holds for a positive proportion of integers A in the range $R < A < 2R$. They also proved in [19] that an asymptotic formula for $P_f(x)$, with f belonging to certain families of quadratic polynomials, holds for x in some range under the assumption of the existence of a Siegel zero for the relevant Dirichlet L-function. The methods used come from a paper of J. B. Friedlander and A. Granville [16] in the study of irregularities in the distributions of primes represented by polynomials. The ideas in [16] originated from the work of H. Maier [31] on irregularities of the distribution of primes in short intervals.

It is note-worthy that certain cases of the asymptotics in (1.1) would follow from a part of another unsolved conjecture due to S. Lang and H. Trotter [29] regarding elliptic curves. To explain the contents of this conjecture, we need some further notation. Let E be be an elliptic curve over $\mathbb{Q}$. If E has good reduction at a prime p (that is, the reduced curve E_p modulo p is non-singular), then a well-known theorem of H. Hasse states that the number of points on E_p differs from $p+1$ by an integer $\lambda_E(p)$ (the trace of the Frobenius morphism of $E/\mathbb{F}_p$) satisfying the bound $|\lambda_E(p)| \leq 2\sqrt{p}$. The Lang-Trotter conjecture predicts an asymptotic formula for the number of primes $p \leq x$ such that $\lambda_E(p)$ equals a fixed integer r. If E has "complex multiplication" and $r \neq 0$, then the primes p satisfying $\lambda_E(p) = r$ lie in quadratic progression. Therefore the Lang-Trotter conjecture is related to the Hardy-Littlewood conjecture stated above. For example, consider the elliptic curve $E : y^2 = x^3 - x$ whose endomorphism ring is isomorphic to $\mathbb{Z}[i]$. It turns out that $p = n^2+1$ for some integer n if and only if $\lambda_E(p) = \pm 2$. See for example [29] for the details.

Conjectures similar to (1.1) also exist for polynomials of higher degree. Hypothesis H of A. Schinzel and W. Sierpiński [37] gives that if f is an irreducible polynomial with integer coefficients that is not congruent to zero modulo any prime, then $f(n)$ is prime for infinitely many integers n. P. T. Bateman and R. A. Horn [10] gave a more explicit version, with an asymptotic formula, of the last-mentioned conjecture.

The following notations and conventions are used throughout paper.

$\mathbb{Z}$, $\mathbb{N}$, $\mathbb{P}$ and $\mathbb{Q}$ denote the sets of integers, natural numbers, primes and rational numbers, respectively.
$f = O(g)$ means $|f| \leq cg$ for some unspecified positive constant c.
$f \ll g$ means $f = O(g)$.
$f \asymp g$ means $c_1 g \leq f \leq c_2 g$ for some unspecified positive constants c_1 and c_2.
$f(x) \sim g(x)$ means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.
$\{x\}$ denotes the fractional part of a real number x.

2. The Conjecture on Average

The von Mangoldt function $\Lambda(n)$, the usual weight with which primes are counted, is defined as follows.

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^l \text{ for some } p \in \mathbb{P} \text{ and } l \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

For the quadratic polynomials of the form $n^2 + k$ for some fixed $k \in \mathbb{N}$ together with the weight of the van Mangoldt function, the conjecture (1.1) takes the following simpler form.

$$\sum_{n \leq x} \Lambda(n^2 + k) \sim \mathfrak{S}(-4k)x. \tag{2.1}$$

The asymptotic formula in (2.1) was studied on average by the authors in [2] and it was established that (2.1) holds true for almost all natural numbers $k \leq K$ if $x^{1+\varepsilon} \leq K \leq x^2/2$. In particular, we have the following.

Theorem 1. *Suppose that $z \geq 3$. Given $B > 0$, we have, for $z^{1/2+\varepsilon} \leq K \leq z/2$,*

$$\sum_{1 \leq k \leq K} \left| \sum_{z < n^2+k \leq 2z} \Lambda(n^2+k) - \mathfrak{S}(-4k) \sum_{z < n^2+k \leq 2z} 1 \right|^2 \ll \frac{Kz}{(\log z)^B}. \tag{2.2}$$

From Theorem 1, the following corollary can be deduced immediately.

Corollary. *Given $A, B > 0$ and $\mathfrak{S}(k)$ as defined above, we have, for $z^{1/2+\varepsilon} \le K \le z/2$, that*

$$\sum_{z<n^2+k\le 2z} \Lambda(n^2+k) = \mathfrak{S}(-4k) \sum_{z<n^2+k\le 2z} 1 + O\left(\frac{\sqrt{z}}{(\log z)^B}\right) \tag{2.3}$$

holds for all natural numbers k not exceeding K with at most $O\left(K(\log z)^{-A}\right)$ exceptions.

It can be easily shown, as done in section 1 of [5], that $\mathfrak{S}(-4k)$ converges and

$$\mathfrak{S}(-4k) \gg \frac{1}{\log k} \gg \frac{1}{\log K} \gg \frac{1}{\log z}.$$

The above inequality shows that the main terms in (2.2) and (2.3) are indeed dominating for the k's under consideration if $B > 1$ and that we truly have an "almost all" result.

Actually, the following sharpened version of Theorem 1 for short segments of quadratic progressions on average was proved in [2].

Theorem 2. *Suppose that $z \ge 3$, $z^{2/3+\varepsilon} \le \Delta \le z^{1-\varepsilon}$ and $z^{1/2+\varepsilon} \le K \le z/2$. Then, given $B > 0$, we have*

$$\int_z^{2z} \sum_{1\le k\le K} \left| \sum_{t<n^2+k\le t+\Delta} \Lambda(n^2+k) - \mathfrak{S}(-4k) \sum_{t<n^2+k\le t+\Delta} 1 \right|^2 \mathrm{d}t \ll \frac{\Delta^2 K}{(\log z)^B}. \tag{2.4}$$

Moreover, we noted in [2] that under the generalized Riemann hypothesis (GRH) for Dirichlet L-functions, the Δ-range in Theorem 2 can be extended to $z^{1/2+\varepsilon} \le \Delta \le z^{1-\varepsilon}$. It is note-worthy that for $\Delta = z^{1/2+\varepsilon}$ the segments of quadratic progressions under consideration are extremely short; that is, they contain only $O(z^\varepsilon)$ elements. Theorem 2 can be interpreted as saying that the asymptotic formula

$$\sum_{t<n^2+k\le t+\Delta} \Lambda(n^2+k) \sim \mathfrak{S}(-4k) \sum_{t<n^2+k\le t+\Delta} 1$$

holds for almost all k and t in the indicated ranges.

These results improve some earlier results of the authors [5] where we used the circle method together with some lemmas in harmonic analysis due to P. X. Gallagher [18] and H. Mikawa [33] and the large sieve for real characters of D. R. Heath-Brown [23]. In [5] k is restricted to be square-free and K can only be in the much smaller range of $z(\log z)^{-A} \le K \le z/2$. Unlike in [5], our approach in the proof of Theorem 2 in [2] is a variant of the dispersion method of J. V. Linnik [30], similar to that used by H. Mikawa in the study of the twin primes problem in [34].

3. Approximating $n^2 + 1$

One may find several results on approximations to the problem of detecting primes of the form n^2+1 in the literature. Note that n^2+1 is a prime if and only if $n+i$ is a Gaussian prime. Hence the problem is equivalent to counting Gaussian primes on the line $\Im z = 1$. Therefore, the problem can be approximated by counting Gaussian primes in narrow strips or sectors which can be studied using Hecke L-functions. In this direction, C. Ankeny [1] and P. Kubilius [27] showed independently that under the Riemann hypothesis for Hecke L-functions for $\mathbb{Q}[i]$ there exist infinitely many Gaussian primes of the form $\pi = m + ni$ with $n < c \log|\pi|$, where c is some positive constant. From this, one infers the infinitude of primes of the form $p = m^2 + n^2$ with $n < c \log p$. Using sieve methods for $\mathbb{Z}[i]$, G. Harman and P. Lewis [21] showed unconditionally that there exist infinitely many primes of the above form with $n \le p^{0.119}$.

Moreover, it is easy to see that n^2+1 represents an infinitude of primes if and only if there are infinitely many primes p such that the fractional part of $\sqrt{p}$ is very small, namely $< 1/\sqrt{p}$. A. Balog, G. Harman and the first-named author [6, 9, 22] dealt with the following related question. Given $0 \le \lambda \le 1$ and a real number θ, for what positive numbers τ can one prove that there exist infinitely many primes p for which the inequality

$$\left\{p^\lambda - \theta\right\} < p^{-\tau}$$

is satisfied? Roughly speaking, three different methods were used to study this problem depending on whether λ lies in the lower, middle or upper part of $[0,1]$. These methods are zero density estimates for the Riemann zeta-function for the lower, approximate functional equation for the Riemann zeta-function for the middle, and estimation of exponential sums over primes for the upper. This problem in turn is related to estimating the number of primes of the form $[n^c]$, where $c > 1$ is fixed and n runs over the positive integers. Primes of this form are referred to as Pyateckiĭ-Šapiro primes [7, 35].

It was established by C. Hooley [24] that if D is not a perfect square then the greatest prime factor of $n^2 - D$ exceeds n^θ infinitely often if $\theta < \theta_0 = 1.1001\cdots$. J.-M. Deshouillers and H. Iwaniec [12] improved this to the effect that n^2+1 has infinitely often a prime factor greater than $n^{\theta_0-\varepsilon}$, where $\theta_0 = 1.202\cdots$ satisfies $2-\theta_0 - 2\log(2-\theta_0) = \frac{5}{4}$. The improvement comes from utilizing mean-value estimates of Kloosterman sums of J.-M. Deshouillers and H. Iwaniec [11]. The result in [12] can also be generalized to $n^2 - D$ by Hooley's arguments.

Moreover, H. Iwaniec [26] also showed that there are infinitely many integers n such that n^2+1 is the product of at most two primes. The result improves a previous one of P. Kuhn [28] that n^2+1 is the product of at most three primes for infinitely many integers n and can be extended to any irreducible polynomial an^2+bn+c with $a > 0$ and c odd.

The results mentioned in the last two paragraphs were based on sieve methods. It is also note-worthy that J. B. Friedlander and H. Iwaniec [17], using results on half-dimensional sieve of H. Iwaniec [25], obtained lower bounds for the number of integers with no small prime divisors represented by a quadratic polynomial.

J. B. Friedlander and H. Iwaniec [15] also proved the celebrated result that there exist infinitely many primes of the form m^2+n^4 (with an asymptotic formula). The set of integers of the form m^2+n^4 contains the set of integers of the form m^2+1 but is still very sparse. The number of such integers not exceeding x is $O(x^{3/4})$. It is generally very difficult to detect primes in sparse sets.

In [3], we approximate the problem of representation of primes by m^2+1 in the following way. For a natural number n let $s(n)$ be the square-free kernel of n; i.e. $s(n) = n/m^2$, where m^2 is the largest square dividing n. We note that $s(n) = 1$ if and only if n is a perfect square. We consider primes of the form $n+1$, where $s(n)$ is small. More precisely, we have the following.

Theorem 3. *Let $\varepsilon > 0$. Then there exist infinitely many primes p such that $s(p-1) \le p^{5/9+\varepsilon}$.*

The set of natural numbers n with $s(n) \le n^{5/9+\varepsilon}$ is also very sparse. More precisely, the number of $n \le x$ with $s(n) \le n^{5/9+\varepsilon}$ is $O(x^{7/9+\varepsilon/2})$ as the following calculation shows.

$$|\{n \le x \ : \ s(n) \le n^{5/9+\varepsilon}\}| \le |\{(a,m) \in \mathbb{N}^2 \ : \ a \le x^{5/9+\varepsilon}, \ am^2 \le x\}| = \sum_{a \le x^{5/9+\varepsilon}} \sum_{m \le \sqrt{x/a}} 1 = O(x^{7/9+\varepsilon/2}).$$

Theorem 3 can be reformulated as follows.

Theorem 3′. *Let $\varepsilon > 0$. Then there exist infinitely many primes of the form $p = am^2+1$ such that $a \le p^{5/9+\varepsilon}$.*

Theorem 3 can be deduced from a Bombieri-Vinogradov type theorem for square moduli, which is as follows.

Theorem 4. *For any $\varepsilon > 0$ and fixed $A > 0$, we have*

$$\sum_{q \le x^{2/9-\varepsilon}} q \max_{\substack{a \\ \gcd(a,q)=1}} \left| \psi(x;q^2,a) - \frac{x}{\varphi(q^2)} \right| \ll \frac{x}{(\log x)^A}, \tag{3.1}$$

where

$$\psi(x;q,a) = \sum_{\substack{n\le x\\ n\equiv a \bmod q}} \Lambda(n)$$

and $\varphi(q)$ *is the number of units in* $\mathbb{Z}/q\mathbb{Z}$.

Theorem 4 improves some results of H. Mikawa and T. P. Peneva [32] and P. D. T. A. Elliott [14]. The key ingredient in the proof of Theorem 4 is the large sieve for square moduli which was studied both independently and jointly by the authors [4, 8, 38].

The classical Bombieri-Vinogradov theorem gives

$$\sum_{q\le \sqrt{x}/(\log x)^{A+5}} \max_{\substack{a\\ \gcd(a,q)=1}} \left|\psi(x;q,a) - \frac{x}{\varphi(q)}\right| \ll \frac{x}{(\log x)^A}.$$

Hence the analogous statement for square moduli should have $q \le x^{1/4}(\log x)^{-A}$ in the sum over q in (3.1). Therefore Theorem 4 is not the complete analogue of the classical theorem. This is due to the fact that in [4] we established a result weaker than the expected analogue of the classical large sieve in the large sieve for square moduli. The latter imperfection is caused by the fact that only a result weaker than the expected was established concerning the spacing of Farey fractions with square denominators. See [4, 8, 38] for the details.

Furthermore, if any of the above-mentioned expectations can be established (spacing of special Farey fractions, large sieve for square moduli or (3.1) with the extended range for q with $q \le x^{1/4-\varepsilon}$), it would follow that there exist infinitely many primes p such that $s(p-1) \le p^{1/2+\varepsilon}$. We can get the same result under the assumption of the generalized Riemann hypothesis for Dirichlet L-functions. We note that the set of n such that $s(n) \le n^{1/2+\varepsilon}$ is "almost" as sparse as the set of numbers m^2+n^4 considered by Friedlander and Iwaniec [15]. Indeed, the number of $n \le x$ such that $s(n) \le n^{1/2+\varepsilon}$ is $O(x^{3/4+\varepsilon/2})$.

It is conceivable that an Elliott-Halberstam [13] type hypothesis holds for primes in arithmetic progressions to square moduli, *i.e.*, that (3.1) holds with the exponent $1/2-\varepsilon$ in place of $2/9-\varepsilon$. This would imply that there exist infinitely many primes p such that $s(p-1) \le p^{\varepsilon}$. A result of this kind comes very close to the conjecture that there exist infinitely many primes of the form n^2+1 since the number of $n \le x$ such that $s(n) \le n^{\varepsilon}$ is $O(x^{1/2+\varepsilon/2})$.

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